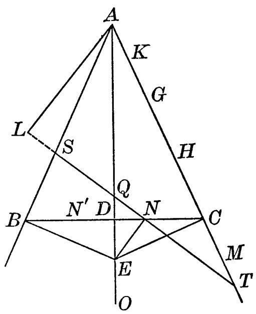
From E as center, with radius CM, describe an arc intersecting BC in the points N and N'. Join EN (or EN'). Through N (or N'), draw ST (or $S'T'$), perpendicular, respectively to EN or EN'. Then $\triangle\ AST$ (or $A'S'T'$) is the required triangle.



Proof.—Draw AL (or AL') perpendicular to ST (or $S'T'$), produced if necessary. Since EN, the perpendicular from E to ST, meets ST in its intersection with BC, it follows that ST is tangent to the parabola. Therefore, the sum of its intercepts on AB and AC is $2k$. That is, $AS + AT = 2k$. Also $AL = R$. For, the $\triangle$'s ALQ and ENQ, both being right $\triangle$'s, and the vertical $\angle$'s at Q being equal, the $\triangle$'s are similar, and

$$AL : EN = AQ : EQ.$$

By composition,

$$(AL + EN) : EN = AE : EQ. \tag{1}$$

But in the right $\triangle\ ENQ$, ND is perpendicular to the hypotenuse EQ. Hence, $EQ = \overline{EN}^2/ED$. Substituting this value of EQ in (1), and dividing the second and fourth terms by EN, we have

$$(AL + EN) : 1 = AE : EN/ED,$$

or

$$(AL + EN)\cdot EN = AE\cdot ED.$$

But in the right $\triangle\ ACE$, $AE\cdot ED = \overline{EC}^2$, and it has been shown that $\overline{EC}^2 = (R + CM)\cdot CM$. Therefore, $(AL + EN)\cdot EN = (R + CM)\cdot CM$. But $EN = CM$ by construction. Therefore $AL = R$.

As indicated in the proof, there are two possible solutions.

Also solved by OSCAR S. ADAMS and N. P. PANDYA.

506. Proposed by S. A. COREY, Albia, Iowa.

Given a pentagon, plane or gauche, whose sides a, b, c, d, e are represented by the vectors x, y, z, v and $(x + y + z + v)$, respectively; and a second pentagon whose sides a_1, b_1, c_1, d_1, e_1 are represented by the vectors r, s, t, u and $(r + s + t + u)$, respectively, where $r = c_1x - c_5c_2y - c_6c_3z + c_5c_6c_4v$, $s = c_2x + c_1y - c_6c_4z - c_6c_3v$, $t = c_1z + c_3x + c_5c_2v + c_5c_4y$, $u = c_1v - c_4x - c_2z + c_3y$; $c_1, c_2, c_3, c_4, c_5, c_6$, being ordinary scalars.

Then prove the existence of the following relation between the sides of the two pentagons:

$$(c_1^2 + c_5c_2^2 + c_6c_3^2 + c_5c_6c_4^2)(x^2 + c_5y^2 + c_6z^2 + c_5c_6v^2) = r^2 + c_5s^2 + c_6t^2 + c_5c_6u^2.$$

SOLUTION BY THE PROPOSER.

Inasmuch as the relation to be established between the vector sides is identical in form with a well-known algebraic identity, it is sufficient to show that the identity holds when certain of the algebraic (scalar) quantities are replaced by vectors. It is well known that the noncommutative character of vector multiplication does not affect the scalar part of the product of linear vector functions in the particular case where that product is a homogeneous *quadratic* function of the vectors employed. As this condition is satisfied in this problem the identity holds, and the existence of the given relation between the sides of the two pentagons is proved. (In forming the vector sides r, s, t, u the direction of the vectors x, y, z, v must be carefully noted.)

Inasmuch as vector *addition* is commutative it follows that the given pentagons may be replaced by certain other pentagons having the same vector sides placed in some other order of succession. Some of the sides may, of course, be zero, in which case the resulting figure would not be a pentagon. To illustrate, let x, y, z be the coterminous vector edges of any parallelepiped, v in this case being zero, and if $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 1$, then r, s, t and u are the diagonals connecting opposite corners of the parallelepiped, and the formula asserts that the sum of the squares of all its 12 edges equals the sum of the squares of its four diagonals; also that the shape of the parallelepiped may be changed without altering the sum of the squares of its diagonals, provided that no change is made in the length of its sides, or in the sum of their squares.

Similarly, if $c_1 = 0$, $v = 0$, $c_2 = c_3 = c_4 = c_5 = c_6 = 1$, the formula asserts that three times the sum of the squares of any three coterminous edges of any parallelepiped equals the sum of

the squares of the three diagonals formed by connecting the extremities of said edges plus the square of the diagonal of the parallelepiped originating at the intersection of said three edges.

Whether the formula could be used in determining the properties of figures in space of four dimensions depends on whether the noncommutative character of vector multiplication does or does not affect the scalar part of the product of linear vector functions, in such space, in the particular case where such product is a homogeneous quadratic function of the vectors employed. If applicable, the number of vectors employed in the formula, and its form, would indicate its usefulness in such higher space.

507. Proposed by A. A. BENNETT, University of Texas.

With the use of the compasses alone construct a circle with area five times as great as that of a given circle. (This problem is said to be due to Napoleon I.)

I. Solution by I. L. Miller, Indiana University.

This problem is equivalent to the construction of $R\sqrt{5}$, where R is the radius of the given circle.

Let A be the center of the given circle. (Fig. 1) Take any point B on its circumference and construct a circle of radius equal to that of the given circle. Let this circle intersect the given circle in C

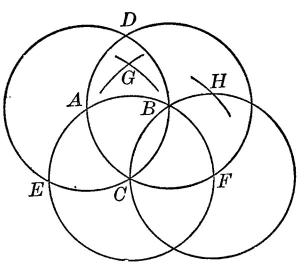


Fig. 1.

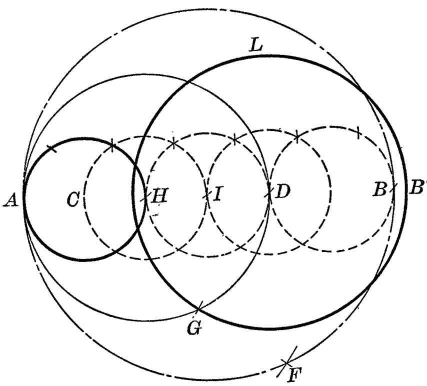


Fig. 2.

and D. Now with C as center construct another circle of radius R, intersecting the first two circles in E and F respectively. And finally with F as center construct another equal circle.

It is evident that $CD = R\sqrt{3}$. With E and F as centers and CD as radius strike arcs, intersecting in G. Then $CG = R\sqrt{2}$. With C as center and CG as radius strike an arc intersecting in H the circumference of the circle last drawn.

Then $EH = R\sqrt{5}$.

This also solves another problem due to Napoleon I; the quadrisection of the circumference of a circle by the use of the compasses alone, which is equivalent to the construction of $R\sqrt{2}$.

II. Solution by Gregory Breit, Student, Johns Hopkins University.

Let AC be the given circle, of radius r. (Fig. 2) From an arbitrary point A on its circumference, lay off three chords of radius length, reaching point H; HA is then a diameter of AC. With H as a center and radius r lay off another circle finding its diameter CI in the same manner as before. Continue this until five such circles are drawn with their centers in the line AB. With H as a center and $2r$ as a radius, lay off the circle AD. With I as center and $3r$ as radius lay off a circle AB. With a radius of length $5r$ cut this circle in F, using A as center. Describe an arc with radius BF from A as a center, cutting circle AD in G. With DG as radius and D as center describe the circle LGB' which is a circle whose area is five times the area of AC. For chord $AF = 5r$, and diameter $AB = 6r$. Hence, chord $BF = \sqrt{36r^2 - 25r^2} = r\sqrt{11}$, chord $AG =$ chord $BF = r\sqrt{11}$, and diameter $AD = 4r$.